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Killing spinors are Killing vector fields in Riemannian supergeometry

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Abstract

A supermanifold M is canonically associated to any pseudo-Riemannian spin manifold (M_0, g_0). Extending the metric g_0 to a field g of bilinear forms $g(p)$ on T_pM, $p \in M_0$, the pseudo-Riemannian supergeometry of (M, g) is formulated as G-structure on M, where G is a supergroup with even part $G_0 \cong \mathrm{Spin}(k, l)$; (k, l) the signature of (M_0, g_0). Killing vector fields on (M, g) are, by definition, infinitesimal automorphisms of this G-structure. For every spinor field s there exists a corresponding odd vector field X_s on M. Our main result is that X_s is a Killing vector field on (M, g) if and only if s is a twistor spinor. In particular, any Killing spinor s defines a Killing vector field X_s.

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1. Introduction to supergeometry

First we introduce the supergeometric language which is needed to formulate the main result of the paper. Standard references on supergeometry are [M,L,K].

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1.1. Supermanifold

We consider pairs $(M_0, \mathcal{A})$, where M_0 is a C^∞-manifold and $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ is a sheaf of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras; $\dim M_0 = m$.

Example 1. We denote by $\mathcal{C}^\infty_{M_0}$ the sheaf of (smooth) functions of M_0. It associates to an open set $U \subset M_0$ the algebra $\mathcal{C}^\infty_{M_0}(U) = C^\infty(U)$ of smooth functions on U. Let E be a (smooth) vector bundle over M_0 and $\mathcal{E}$ the corresponding locally free sheaf of $\mathcal{C}^\infty_{M_0}$-modules: $\mathcal{E}$ associates to an open set $U \subset M_0$ the $C^\infty(U)$-module $\mathcal{E}(U) = \Gamma(U, E)$ of sections of E over U. Conversely, any locally free sheaf $\mathcal{E}$ of $\mathcal{C}^\infty_{M_0}$-modules defines a vector bundle $E \to M_0$. The exterior sheaf $\wedge\mathcal{E} = \wedge^{\mathrm{ev}}\mathcal{E} + \wedge^{\mathrm{odd}}\mathcal{E}$ is a sheaf of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras on M_0.

Definition 1. The pair $M = (M_0, \mathcal{A})$ is called a (differentiable) supermanifold of dimension $m|n$ over M_0 if for all $p \in M_0$ there exists an open neighborhood $U \ni p$ and a rank n free sheaf $\mathcal{E}_U$ of $\mathcal{C}^\infty_U$-modules over U such that $\mathcal{A}|_U \cong \wedge\mathcal{E}_U$ (as sheaves of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras). The (local) sections of $\mathcal{A}$ are called (local) functions on M.

From Definition 1 it follows that there exists a canonical epimorphism $\epsilon : \mathcal{A} \to \mathcal{C}^\infty_{M_0}$, which is called the *evaluation map*. Its kernel is the ideal $\mathcal{J}$ generated by $\mathcal{A}_1$: $\ker\epsilon = \mathcal{J} = \langle\mathcal{A}_1\rangle = \mathcal{A}_1 + \mathcal{A}_1^2$. By the construction of Example 1 to any vector bundle $E \to M_0$ we have associated a supermanifold $M(E) = (M_0, \mathcal{A} = \wedge\mathcal{E})$. In this case the exact sequence

$$0 \to \mathcal{J} = \langle\mathcal{E}\rangle \to \mathcal{A} = \wedge\mathcal{E} \xrightarrow{\epsilon} \mathcal{C}^\infty_{M_0} \to 0$$

of sheafs of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras has a canonical splitting $\mathcal{C}^\infty_{M_0} \hookrightarrow \wedge\mathcal{E} = \mathcal{C}^\infty_{M_0} + \langle\mathcal{E}\rangle$.

Let $(x^1, \ldots, x^m)$ be local coordinates for M_0 defined on an open set $U \subset M_0$ such that $\mathcal{A}|_U \cong \wedge\mathcal{E}_U$, where $\mathcal{E}_U$ is a rank n free sheaf of $\mathcal{C}^\infty_U$-modules, cf. Definition 1. Let $\theta_1, \ldots, \theta_n$ be sections of $\mathcal{E}_U$ trivializing the vector bundle E_U associated to the sheaf $\mathcal{E}_U$. Note that $x^1, \ldots, x^m, \theta_1, \ldots, \theta_n$ can be considered as local functions on the supermanifold M. Moreover, any local function $f \in \mathcal{A}(U)$ is of the form

$$f = \sum_{\alpha\in\mathbb{Z}_2^n} f_\alpha(x^1, \ldots, x^m)\theta^\alpha, \quad f_\alpha(x^1, \ldots, x^m) \in C^\infty(U) = \mathcal{C}^\infty_{M_0}(U), \tag{1}$$

where $\theta^\alpha := \theta_1^{\alpha_1} \wedge \cdots \wedge \theta_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Definition 2. The tuple $(x^i, \theta_j) = (x^1, \ldots, x^m, \theta_1, \ldots, \theta_n)$ is called a *local coordinate system* for M over U.

The evaluation map applied to a (local) function $f = f(x^1, \ldots, x^m, \theta_1, \ldots, \theta_n)$ with expansion (1) is given by

$$\epsilon(f) = f(x^1, \ldots, x^m, 0, \ldots, 0) = f_{(0,\ldots,0)}(x^1, \ldots, x^m).$$

Let $M = (M_0, \mathcal{A})$ and $N = (N_0, \mathcal{B})$ be supermanifolds.

Definition 3. A *morphism* $\Phi : M \to N$ is a pair $\Phi = (\varphi, \phi)$, where $\varphi : M_0 \to N_0$ is a differentiable map and $\phi : \mathcal{B} \to \varphi_* \mathcal{A}$ is a morphism of sheaves of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras. Φ is called an *isomorphism* if φ is a diffeomorphism and ϕ is an isomorphism. An isomorphism $\Phi : M \to M$ is called *automorphism* of M.

In local coordinate systems (x^i, θ_j) for M and $(\tilde{x}^k, \tilde{\theta}_l)$ for N a morphism Φ is expressed by p even functions $\tilde{x}^k(x^1, \dots, x^m, \theta_1, \dots, \theta_n)$, $k = 1, \dots, p$, and q odd functions $\tilde{\theta}_l(x^1, \dots, x^m, \theta_1, \dots, \theta_n)$, $l = 1, \dots, q$; where $(p, q) = \dim N$.

1.2. Tangent vector/vector field

Let $M = (M_0, \mathcal{A})$ be a supermanifold. For any point $p \in M_0$ the evaluation map $\epsilon : \mathcal{A} \to C^\infty_{M_0}$ induces an epimorphism $\epsilon_p : \mathcal{A}_p \to \mathbb{R}$, $\epsilon_p(f) := \epsilon(f)(p)$, where $\mathcal{A}_p$ denotes the stalk of $\mathcal{A}$ at p. For $\alpha \in \mathbb{Z}_2 = \{0, 1\}$ we define

$$(T_pM)_\alpha := \{v : \mathcal{A}_p \to \mathbb{R}, \quad \mathbb{R}\text{-linear} \mid v(fg) = v(f)\epsilon_p(g) + (-1)^{\alpha\tilde{f}}\epsilon_p(f)v(g)\},$$

where the equation is required for all $f, g \in \mathcal{A}_p$ of pure degree and $\tilde{f} \in \{0, 1\}$ denotes the degree of f.

Definition 4. The *tangent space* of M at $p \in M_0$ is the $\mathbb{Z}_2$-graded vector space $T_pM = (T_pM)_0 + (T_pM)_1$. The elements of T_pM are called *tangent vectors*. Any morphism $\Phi = (\varphi, \phi) : M = (M_0, \mathcal{A}) \to N = (N_0, \mathcal{B})$ induces linear maps $d\Phi(p) : T_pM \to T_{\varphi(p)}N$, defined by $(\mathrm{d}\Phi(p)v)(f) := v(\phi_p(f))$, $p \in M_0$, $v \in T_pM$, $f \in \mathcal{B}_{\varphi(p)}$, where $\phi_p : \mathcal{B}_{\varphi(p)} \to \mathcal{A}_p$ is the morphism of stalks associated to $\phi : \mathcal{B} \to \varphi_*\mathcal{A}$. The map $\mathrm{d}\Phi(p)$ is called the *differential at* p of Φ.

The sheaf $\mathrm{Der}\,\mathcal{A}$ of derivations of $\mathcal{A}$ over $\mathbb{R}$ is a sheaf of $\mathbb{Z}_2$-graded $\mathcal{A}$-modules: $\mathrm{Der}\,\mathcal{A} = (\mathrm{Der}\,\mathcal{A})_0 + (\mathrm{Der}\,\mathcal{A})_1$, where

$$(\mathrm{Der}\,\mathcal{A})_\alpha = \{X : \mathcal{A} \to \mathcal{A}, \quad \mathbb{R}\text{-linear} \mid X(fg) = X(f)g + (-1)^{\alpha\tilde{f}} fX(g)\},$$

where the equation is required for all $f, g \in \mathcal{A}$ of pure degree.

Definition 5. The sheaf $\mathcal{T}_M = \mathrm{Der}\,\mathcal{A}$ is called the *tangent sheaf* of $M = (M_0, \mathcal{A})$. The sections of $\mathcal{T}_M$ are called *vector fields*.

Any local coordinate system (x^i, θ_j) over U gives rise to even vector fields $\partial/\partial x^i$ and odd vector fields $\partial/\partial\theta_j$ over U. The action of the vector fields $\partial/\partial x^i$, $\partial/\partial\theta_j$ on a function f with expansion (1) is given by

$$\frac{\partial f}{\partial x^i} = \sum_\alpha \frac{\partial f_\alpha(x^1, \dots, x^m)}{\partial x^i}\theta^\alpha,$$

$$\frac{\partial f}{\partial \theta_j} = \sum_\alpha \alpha_j(-1)^{\alpha_1+\cdots+\alpha_{j-1}} f_\alpha(x^1, \dots, x^m)\theta_1^{\alpha_1} \wedge \cdots \wedge \theta_j^{\alpha_j - 1} \wedge \cdots \wedge \theta_n^{\alpha_n}.$$

Any vector field X on M over U can be written as

$$X = \sum_{i=1}^{m} X^i(x^1, \ldots, x^m, \theta_1, \ldots, \theta_n)\frac{\partial}{\partial x^i} + \sum_{j=1}^{n} Y^j(x^1, \ldots, x^m, \theta_1, \ldots, \theta_n)\frac{\partial}{\partial \theta_j},$$

where $X^i, Y^j \in \mathcal{A}(U)$.

If $\Phi = (\varphi, \phi) : M = (M_0, \mathcal{A}) \to N = (N_0, \mathcal{B})$ is an isomorphism then φ^{-1} and $\phi^{-1} : \varphi_*\mathcal{A} \to \mathcal{B}$ exist and give rise to an isomorphism $\mathcal{A} \to \varphi_*^{-1}\mathcal{B}$. The induced isomorphism between the corresponding sheaves of derivations is denoted by

$$\mathrm{d}\Phi : \mathcal{T}_M \to \varphi_*^{-1}\mathcal{T}_N$$

and is called the *differential* of Φ. For any open $U \subset M_0$ the differential $\mathrm{d}\Phi$ is expressed by an $\mathcal{A}(U)$-linear map $d\Phi_U : \mathcal{T}_M(U) \to \mathcal{T}_N(\varphi(U))$, where the action of $\mathcal{A}(U)$ on $\mathcal{T}_N(\varphi(U))$ is defined using the isomorphism $\mathcal{A}(U) \xrightarrow{\sim} \mathcal{B}(\varphi(U))$ induced by ϕ^{-1}.

Let X be a vector field defined on some open set $U \subset M_0$ and $p \in U$. Then we can define the *value* $X(p) \in T_pM$ of X at p,

$$X(p)(f) := \epsilon_p(X(f)), \quad f \in \mathcal{A}_p.$$

However, unless $\dim M = m|n = m|0$, a vector field is not determined by its values.

Finally, we relate the tangent spaces and tangent sheaves of M and M_0. Any even tangent vector $v \in (T_pM)_0$ annihilates the ideal $\mathcal{J} = \ker \epsilon$ in the exact sequence

$$0 \to \mathcal{J} \to \mathcal{A} \xrightarrow{\epsilon} \mathcal{C}^\infty_{M_0} \to 0 \tag{2}$$

and hence defines a tangent vector to M_0. More explicitly, we define a map $\epsilon : T_pM \to T_pM_0$ by the equation

$$\epsilon(v)(\epsilon(f)) = v_0(f),$$

where $v = v_0 + v_1 \in (T_pM)_0 + (T_pM)_1$, $f \in \mathcal{A}_p$ and $f \mapsto \epsilon(f)$ is the evaluation map of stalks $\epsilon : \mathcal{A}_p \to (\mathcal{C}^\infty_{M_0})_p$.

Proposition 1. *There is a canonical exact sequence of $\mathbb{Z}_2$-graded vector spaces*

$$0 \to (T_pM)_1 \to T_pM \xrightarrow{\epsilon} T_pM_0 \to 0.$$

In particular, ϵ induces a canonical isomorphism $(T_pM)_0 \xrightarrow{\sim} T_pM_0$.

Similarly, on the level of tangent sheaves we define $\epsilon : \mathcal{T}_M \to \mathcal{T}_{M_0}$ by the equation

$$\epsilon(X)(\epsilon(f)) = \epsilon(X_0(f)),$$

where $X = X_0 + X_1 \in (\mathcal{T}_M(U))_0 + (\mathcal{T}_M(U))_1$, $f \in \mathcal{A}(U)$ and $U \subset M_0$ open.

Proposition 2. *There is a canonical exact sequence of sheaves of $\mathcal{A}$-modules*

$$0 \to \ker \epsilon \to \mathcal{T}_M \xrightarrow{\epsilon} \mathcal{T}_{M_0} \to 0, \tag{3}$$

where $\ker\epsilon = (\mathcal{T}_M)_1 + \mathcal{J}\mathcal{T}_M$. *In particular, there is the following exact sequence of* $\mathcal{A}$-*modules:*

$$0 \to (\mathcal{J}\mathcal{T}_M)_0 \to (\mathcal{T}_M)_0 \to \mathcal{T}_{M_0} \to 0.$$

1.3. Frame/frame field/local coordinates

Definition 6. Let $V = V_0 + V_1$ be a $\mathbb{Z}_2$-graded vector space of *rank* $m|n$, i.e. $\dim V_0 = m$ and $\dim V_1 = n$. A *basis* of V is a tuple $(b_1, \dots, b_{m+n})$ such that $(b_1, \dots, b_m)$ is a basis of V_0 and $(b_{m+1}, \dots, b_{m+n})$ is a basis of V_1. Let $M = (M_0, \mathcal{A})$ be a supermanifold and $p \in M_0$. A *frame* at p is a basis of T_pM. A tuple $(X_1, \dots, X_{m+n})$ of vector fields defined on an open subset $U \subset M_0$ is called a *frame field* if $(X_1(p), \dots, X_{m+n}(p))$ is a frame at all points $p \in U$. We denote by $\mathcal{F}(U)$ the set of all frame fields over U. The sheaf of sets $U \mapsto \mathcal{F}(U)$ is called the *sheaf of frame fields*.

Any local coordinate system (x^i, θ_j) over U gives rise to the frame field $(\partial/\partial x^i, \partial/\partial\theta_j)$ over U.

1.4. Supergroup

Let $\boldsymbol{A} = \boldsymbol{A}_0 + \boldsymbol{A}_1$ be an associative $\mathbb{Z}_2$-graded $\mathbb{R}$-algebra with unit. We will always assume that $\boldsymbol{A}$ is *supercommutative*, i.e. $ab = (-1)^{\tilde{a}\tilde{b}}ba$ for all $a, b \in \boldsymbol{A}_0 \cup \boldsymbol{A}_1$. Under this assumption any left-$\boldsymbol{A}$-module carries a canonical right-$\boldsymbol{A}$-module structure and vice versa; so we will simply speak of $\boldsymbol{A}$-modules. For any supermanifold $M = (M_0, \mathcal{A})$ the algebra of functions $\mathcal{A}(M_0)$ is supercommutative, associative and has a unit.

For any set Σ and non-negative integers r, s we denote by $\mathrm{Mat}(r, s, \Sigma)$ the set of $r \times s$-matrices with entries in Σ and put $\mathrm{Mat}(r, \Sigma) := \mathrm{Mat}(r, r, \Sigma)$. Any partition $(r = m + n, s = k + l)$ defines a $\mathbb{Z}_2$-grading on the $\boldsymbol{A}$-module $V = \mathrm{Mat}(r, s, \mathrm{A})$:

$$V_0 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \; A \in \mathrm{Mat}(m, k, \boldsymbol{A}_0), D \in \mathrm{Mat}(n, l, \boldsymbol{A}_0), \right.$$
$$\left. B \in \mathrm{Mat}(m, l, \boldsymbol{A}_1), C \in \mathrm{Mat}(n, k, \boldsymbol{A}_1) \right\},$$

$$V_1 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \; A \in \mathrm{Mat}(m, k, \boldsymbol{A}_1), D \in \mathrm{Mat}(n, l, \boldsymbol{A}_1), \right.$$
$$\left. B \in \mathrm{Mat}(m, l, \boldsymbol{A}_0), C \in \mathrm{Mat}(n, k, \boldsymbol{A}_0) \right\}.$$

The $\mathbb{Z}_2$-graded $\boldsymbol{A}$-module $V = V_0 + V_1$ is denoted by $\mathrm{Mat}(m|n, k|l, \boldsymbol{A})$. Matrix multiplication turns $\mathrm{Mat}(m|n, \boldsymbol{A}) := \mathrm{Mat}(m|n, m|n, \boldsymbol{A})$ into an associative $\mathbb{Z}_2$-graded algebra with unit.

Definition 7. A *super Lie bracket* on a $\mathbb{Z}_2$-graded vector space $V = V_0 + V_1$ is a bilinear map $[\cdot,\cdot] : V \times V \to V$ such that for all $x, y, z \in V_0 \cup V_1$ we have:
(i) $\widetilde{[x, y]} = \tilde{x} + \tilde{y}$,
(ii) $[x, y] = -(-1)^{\tilde{x}\tilde{y}}[y, x]$ and
(iii) $[x, [y, z]] = [[x, y], z] + (-1)^{\tilde{x}\tilde{y}}[y, [x, z]]$.
The pair $(V, [\cdot,\cdot])$ is called a *super Lie algebra*.

The supercommutator

$$[X, Y] = XY - (-1)^{\tilde{X}\tilde{Y}} YX, \qquad X, Y \in \mathrm{Mat}(m|n, A)_0 \cup \mathrm{Mat}(m|n, A)_1,$$

defines a super Lie bracket on the $\mathbb{Z}_2$-graded vector space $\mathrm{Mat}(m|n, A)$. The super Lie algebra $(\mathrm{Mat}(m|n, A), [\cdot,\cdot])$ is denoted by $\mathfrak{gl}_{m|n}(A)$. We put

$$GL_{m|n}(A) := \{g \in \mathrm{Mat}(m|n, A)_0 | g \text{ is invertible}\}.$$

Similarly, if V is a $\mathbb{Z}_2$-graded A-module $\mathrm{End}_A(V)$ carries a canonical super Lie algebra structure, which is denoted by $\mathfrak{gl}_A(V)$. By definition $GL_A(V)$ is the group of invertible elements of $\mathrm{End}_A(V)$. Finally, we will use the convention $\mathfrak{gl}_{m|n} := \mathfrak{gl}_{m|n}(\mathbb{R})$, $\mathfrak{gl}(V) := \mathfrak{gl}_{\mathbb{R}}(V)$, $GL(V) := GL_{\mathbb{R}}(V)$.

Definition 8. A *supergroup* G is a contravariant functor $M \mapsto G(M)$ from the category of supermanifolds into the category of groups. Let H, G be supergroups. We say that H is a super subgroup of G and write $H \subset G$ if $H(M) \subset G(M)$ is a subgroup and $H(\Phi) = G(\Phi)|H(N)$ for all supermanifolds M, N and morphisms $\Phi : M \to N$.

Example 2. The *general linear supergroup* $GL_{m|n}$ is the supergroup $M \to GL_{m|n}(M)$ obtained as composition of the following two functors:
(i) the contravariant functor $M = (M_0, \mathcal{A}) \to \mathcal{A}(M_0)$ from the category of supermanifolds into that of asssociative, supercommutative algebras with unit,
(ii) the covariant functor $A \to GL_{m|n}(A)$ from the category of associative, supercommutative algebras with unit into that of groups.

Definition 9. A *linear super Lie algebra* $\mathfrak{g}$ is a super Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}_{m|n}$ (for some $m|n$). A *linear supergroup* is a super subgroup $G \subset GL_{m|n}$ (for some $m|n$).

Example 3. Let $\mathfrak{g} \subset \mathfrak{gl}_{m|n}$ be a linear super Lie algebra. For any associative, supercommutative algebra with unit A we can consider the super Lie algebra $\mathfrak{g} \otimes A \subset \mathfrak{gl}_{m|n}(A)$. Its even part $\mathfrak{g}(A) := (\mathfrak{g} \otimes A)_0$ is a Lie algebra. If $A = \mathcal{A}(M_0)$ is the algebra of functions of a supermanifold $M = (M_0, \mathcal{A})$ then it is easy to see that the exponential series

$$\sum_{n=0}^{\infty} \frac{1}{n!} X^n, \qquad X \in \mathrm{Mat}(m|n, A),$$

converges (locally uniformly) to an element $\exp X \in GL_{m|n}(A)$. Now let $G(A)$ be the subgroup of $GL_{m|n}(A)$ generated by $\exp \mathfrak{g}(A)$. Then the functor $M = (M_0, \mathcal{A}) \mapsto G(M) := G(\mathcal{A}(M_0))$ is a linear supergroup, which we denote by $\exp \mathfrak{g}$.

1.5. G-structure

Let $M = (M_0, \mathcal{A})$ be a supermanifold of $\dim M = m|n$. For any open subset $U \subset M_0$ we consider the supermanifold $M|_U := (U, \mathcal{A}|_U)$. The general linear supergroup $GL_{m|n}$ induces a sheaf $\mathcal{GL}_M$ of groups over M_0: $\mathcal{GL}_M(U) := GL_{m|n}(M|_U) = GL_{m|n}(\mathcal{A}(U))$, $U \subset M_0$ open. The group $\mathcal{GL}_M(U)$ acts naturally (from the right) on the set $\mathcal{F}(U)$ of frame fields over U. This action turns $\mathcal{F}$ into a sheaf of $\mathcal{GL}_M$-sets. Now let $G \subset GL_{m|n}$ be a linear supergroup and $\mathcal{G}$ the corresponding sheaf of groups, i.e. $\mathcal{G}(U) = G(M|_U)$ for all open $U \subset M_0$. Since $\mathcal{G}$ is a sheaf of subgroups $\mathcal{G} \subset \mathcal{GL}_M$ the sheaf $\mathcal{F}$ of frame fields of M is, in particular, a sheaf of $\mathcal{G}$-sets.

Definition 10. Let $M = (M_0, \mathcal{A})$, $\dim M = m|n$, be a supermanifold and $G \subset GL_{m|n}$ a linear supergroup. A *G-structure* on M is a sheaf $\mathcal{F}_G$ of $\mathcal{G}$-subsets $\mathcal{F}_G \subset \mathcal{F}$ such that for all $p \in M_0$ there exists an open neighborhood $U \ni p$ for which $\mathcal{G}(U)$ acts simply transitively on $\mathcal{F}_G(U)$.

Example 4. For any supermanifold M, $\dim M = m|n$, the sheaf of frame fields $\mathcal{F}$ is a $GL_{m|n}$-structure.

1.6. Automorphism of G-structure

We denote by $\mathrm{Aut}(M)$ the group of all automorphisms of the supermanifold M, see Definition 3. The differential $\mathrm{d}\Phi : \mathcal{T}_M \to \varphi_*^{-1}\mathcal{T}_M$ of any $\Phi = (\varphi, \phi) \in \mathrm{Aut}(M)$ induces an isomorphism $\mathcal{F} \to \varphi_*^{-1}\mathcal{F}$, again denoted by $\mathrm{d}\Phi$. Now let $\mathcal{F}_G \subset \mathcal{F}$ be a G-structure on M, for some linear supergroup $G \subset GL_{m|n}$. For simplicity we can assume that $G = \exp \mathfrak{g}$ as in Example 3.

Definition 11. $\Phi = (\varphi, \phi) \in \mathrm{Aut}(M)$ is called an *automorphism* of the G-structure $\mathcal{F}_G$ if $\mathrm{d}\Phi\mathcal{F}_G \subset \varphi_*^{-1}\mathcal{F}_G$.

We recall that any $p \in M_0$ has an open neighborhood U such that $\mathcal{G}(U)$ acts simply transitively on $\mathcal{F}_G(U)$. Such open sets $U \subset M_0$ will be called *small*. If $U \subset M_0$ is small then $\mathcal{F}_G(U) = E\mathcal{G}(U)$ for any frame field $E \in \mathcal{F}_G(U)$. Here the right-action of the group $\mathcal{G}(U)$ on $\mathcal{F}_G(U)$ is simply denoted by juxtaposition.

Proposition 3. *$\Phi \in \mathrm{Aut}(M)$ is an automorphism of the G-structure $\mathcal{F}_G$ iff*

$$\mathrm{d}\Phi_{U'} E|_{U'} \in E|_{\varphi(U')}\mathcal{G}(\varphi(U'))$$

for all small $U \subset M_0$, $E \in \mathcal{F}_G(U)$ and open $U' \subset U$ such that $\varphi(U') \subset U$.

For any open set $U \subset M_0$ the vector space $\mathcal{T}_M(U)^{m+n}$ of $(m+n)$-tuples of vector fields is naturally a right-module of the associative, $\mathbb{Z}_2$-graded algebra $\mathrm{Mat}(m|n, \mathcal{A}(U))$. In particular, it is a right-module of the super Lie algebra $\mathfrak{g}\otimes\mathcal{A}(U) \subset \mathfrak{g}_{m|n}(\mathcal{A}(U))$. On the other hand, $\mathcal{T}_M(U)$ (and hence $\mathcal{T}_M(U)^{m+n}$) is naturally a left-module for the super Lie algebra $\mathcal{T}_M(U)$ of local vector fields. The action on $\mathcal{T}_M(U)$ is given by the adjoint representation, i.e. by the supercommutator $\mathrm{ad}_X Y = X \circ Y - (-1)^{\tilde{X}\tilde{Y}} Y \circ X$, $X, Y \in \mathcal{T}_M(U)$ of pure degree. The corresponding action on $\mathcal{T}_M(U)^{m+n}$ is denoted by L_X ("Lie derivative"):

$$L_X E := ([X, X_1], \dots, [X, X_{m+n}]), \quad E = (X_1, \dots, X_{m+n}) \in \mathcal{T}_M(U)^{m+n}.$$

Proposition 3 motivates the following definition.

Definition 12. A vector field X on M is an *infinitesimal automorphism* of the G-structure $\mathcal{F}_G$ if

$$L_{X|_U} E \in E(\mathfrak{g} \otimes \mathcal{A}(U))$$

for all small $U \subset M_0$, $E \in \mathcal{F}_G(U)$.

2. Supergeometry associated to the spinor bundle

2.1. The supermanifold M(S)

Let (M_0, g_0) be a (smooth) pseudo-Riemannian spinmanifold with spinor bundle $S \to M_0$. The corresponding locally free sheaf of $\mathcal{C}^\infty_{M_0}$-modules will be denoted by $\mathcal{S}$; $\mathcal{S}(U) = \Gamma(U, S)$, $U \subset M_0$ open. To the vector bundle $S \to M_0$ we associate the supermanifold $M : M(S) = (M_0, \mathcal{A} = \wedge\mathcal{S})$.

Consider the $\mathbb{Z}_2$-graded vector bundle $TM_0 + S^* \to M_0$ with even part TM_0 and odd part S^*.

Proposition 4. *For any $p \in M_0$ there is a canonical isomorphism of $\mathbb{Z}_2$-graded vector spaces $\iota_p : T_pM_0 + S^*_p \xrightarrow{\sim} T_pM$.*

Proof. We define $\iota_p^{-1}|(T_pM)_0 := \epsilon|(T_pM)_0$, see Proposition 1. Now it is sufficient to construct a canonical isomorphism $S^* \xrightarrow{\sim} (T_pM)_1$. For any section $s \in \Gamma(U, S^*)$ interior multiplication $\iota(s)$ by s defines an odd derivation of the $\mathbb{Z}_2$-graded algebra $\mathcal{A}(U) = \Gamma(U, \wedge S)$, i.e. a vector field $X_s := \iota(s) \in \mathcal{T}_M(U)_1$. The value $X_s(p) \in (T_pM)_1$ depends only on $s(p) \in S^*_p$ and we can define $\iota_p(s(p)) := X_s(p)$. □

Using the embedding $\mathcal{C}^\infty_{M_0} \hookrightarrow \wedge\mathcal{S}$, we can consider $\mathcal{T}_M$ as a sheaf of $\mathcal{C}^\infty_{M_0}$-modules. Interior multiplication $s \mapsto \iota(s) = X_s$ defines a monomorphism $\mathcal{S}^* \hookrightarrow (\mathcal{T}_M)_1$ of sheaves of $\mathcal{C}^\infty_{M_0}$-modules. We want to extend this map to $\iota : \mathcal{T}_{M_0} + \mathcal{S}^* \to \mathcal{T}_M$. For a local vector field $X \in \mathcal{T}_{M_0}(U)$ on M_0 we put

$$\iota(X) := \nabla_X \in \mathcal{T}_M(U)_0,$$

where ∇ is the canonical connection on $\wedge S$, i.e. the one induced by the Levi-Civita connection on (M_0, g_0).

Proposition 5. *The map $\iota : \mathcal{T}_{M_0} + \mathcal{S}^* \to \mathcal{T}_M$ is a monomorphism of sheaves of $\mathbb{Z}_2$-graded $C^\infty_{M_0}$-modules. Moreover, $\iota|\mathcal{T}_{M_0}$ defines a splitting of the sequence* (3), *i.e.* $\epsilon \circ \iota|\mathcal{T}_{M_0} = \mathrm{id}$.

Note that given any vector bundle E and connection D on E we can canonically define $\iota_{E,D} : \mathcal{T}_{M_0} + \mathcal{E}^* \hookrightarrow \mathcal{T}_M$, where $M = M(E)$ and $\mathcal{E}$ is the sheaf of local sections of E. In Proposition 5 we have $\iota = \iota_{S,\nabla}$.

2.2. The coadjoint representation of the Poincaré super Lie algebras

Let $(V_0, \langle\cdot,\cdot\rangle)$ be a pseudo-Euclidean vector space of signature (k, l), $k + l = m$, and V_1 the spinor module of the group $\mathrm{Spin}(V_0)$, $n := \dim V_1$. Put $V := V_0 + V_1$. The vector space $\mathfrak{p}(V) := \mathfrak{spin}(V_0) + V$ carries the structure of $\mathfrak{spin}(V_0)$-module. We want to extend this structure to a super Lie bracket $[\cdot,\cdot]$ on $\mathfrak{p}(V)$ which satisfies $[V_0, V] = 0$ and $[V_1, V_1] \subset V_0$. Such an extension is precisely given by a $\mathrm{Spin}(V_0)$-equivariant map $\pi : \vee^2 V_1 \to V_0$; here $\vee^2$ denotes the symmetric square.

Definition 13. The structure of super Lie algebra defined on $\mathfrak{p}(V)$ by the map π is called a *Poincaré super Lie algebra*.

We denote by $\rho : V_0 \to \mathrm{End}(V_1)$ the (standard) Clifford multiplication.

Definition 14. A bilinear form β on the spinor module is called *admissible* if:
(1) β is symmetric or skew symmetric. We define the symmetry σ of β to be $\sigma(\beta) = +1$ in the first case and $\sigma(\beta) = -1$ in the second.
(2) Clifford multiplication $\rho(v)$, $v \in V_0$, is either symmetric or skew symmetric. Accordingly, we define the type τ of β to be $\tau(\beta) = \pm 1$.
An admissible form β is called *suitable* if $\sigma(\beta)\tau(\beta) = +1$.

Given a suitable bilinear form β on V_1 we define $\pi = \pi_{\rho,\beta} : \vee^2 V_1 \to V_0$ by

$$\langle \pi(s_1 \vee s_2), v\rangle = \beta(\rho(v)s_1, s_2), \quad s_1, s_2 \in V_1, \quad v \in V_0. \tag{4}$$

The map π is $\mathrm{Spin}(V_0)$-equivariant. Hence it defines on the vector space $\mathfrak{p}(V)$ the structure of Poincaré super Lie algebra. The following theorem was proved in [AC]:

Theorem 1. *Any Spin(V_0)-equivariant map $\vee^2 V_1 \to V_0$ is a linear combination of maps π_{ρ,β_i}, β_i suitable.*

All admissible bilinear forms on the spinor module were explicitly determined in [AC]. The spinor module carries a *non-degenerate* suitable bilinear form β for all values of $m = k + l$ and $s = k - l$ except for $(m, s) = (5, 7), (6, 0), (6, 6)$ and $(7, 7)$ (mod 8, mod 8). Now we assume that a non-degenerate suitable bilinear form β on V_1 is given. The map $\pi = \pi_{\rho,\beta}$ defines on $\mathfrak{p}(V)$ the structure of Poincaré super Lie algebra such that $[V_1, V_1] = V_0$.

Given a super Lie algebra $\mathfrak{g}$ the *coadjoint representation* $\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$, $x \mapsto \mathrm{ad}^*_x$, is defined by the equation

$$\mathrm{ad}^*_x(y^*) = -(-1)^{\tilde{x}\tilde{y^*}} y^* \circ \mathrm{ad}_x,$$

for $x \in \mathfrak{g}$ and $y^* \in \mathfrak{g}^*$ of pure degree.

Proposition 6. *The coadjoint representation of* $\mathfrak{p}(V)$ *preserves the subspace* $V^\perp = \{x^* \in \mathfrak{p}(V)^* | x^*(V) = 0\} \subset \mathfrak{p}(V)^*$ *and hence induces a representation* $\alpha : \mathfrak{p}(V) \to \mathfrak{gl}(V^*)$ *on* $V^* \cong \mathfrak{p}(V)^*/V^\perp$. *It has kernel* $\ker \alpha = V_0$ *and therefore induces a faithful representation of the super Lie algebra* $\mathfrak{p}(V)/V_0$ *on* V^*.

Once we choose a basis $b = (b_1, \ldots, b_{m+n})$ of V^*, we can identify $\alpha(\mathfrak{p}(V)) \subset \mathfrak{gl}(V^*)$ with a subalgebra $\alpha(\mathfrak{p}(V))^b \subset \mathfrak{gl}_{m|n}$, where $A \mapsto A^b$ denotes the isomorphism $\mathfrak{gl}(V^*) \to \mathfrak{gl}_{m|n}$ defined by b. If moreover $(b_1, \ldots, b_m)$ is an orthonormal basis of $V_1^\perp \cong V_0^*$ then the even part $\alpha(\mathfrak{p}(V))^b_0 \cong \mathfrak{spin}(k, l)$ is a canonically embedded spinor Lie algebra, i.e.

$$\alpha(\mathfrak{p}(V))^b_0 = \mathfrak{spin}_\sigma := \left\{ \begin{pmatrix} A & 0 \\ 0 & \sigma(A) \end{pmatrix} \middle| A \in \mathfrak{so}(k, l) \subset \mathfrak{gl}_m \right\},$$

where $\sigma : \mathfrak{so}(k, l) \to \mathfrak{gl}_n$ is equivalent to the spinor representation.

The linear group $\mathrm{Spin}_\sigma \subset GL_{m|n}(\mathbb{R})$ generated by the Lie algebra $\mathfrak{spin}_\sigma \subset (\mathfrak{gl}_{m|n})_0 \cong \mathfrak{gl}_m \oplus \mathfrak{gl}_n$ acts on the set of bases of V^* from the right.

Proposition 7. *Assume that* $\alpha(\mathfrak{p}(V))^b_0 = \mathfrak{spin}_\sigma$ *and* $b' = bg$ *for some* $g \in \mathit{Spin}_\sigma$. *Then* $\alpha(\mathfrak{p}(V))^b = \alpha(\mathfrak{p}(V))^{b'}$.

Proof. This follows from the fact that $\alpha(\mathfrak{p}(V))^b_0 = \mathfrak{spin}_\sigma$ and $\alpha(\mathfrak{p}(V))^b_1 = \alpha(V_1)^b$ are invariant under $\mathfrak{spin}_\sigma = \alpha(\mathfrak{spin}(V_0))^b$. □

Now let $(e_1, \ldots, e_m)$ be an orthonormal basis of V_0 and $(\theta^1, \ldots, \theta^n)$ a basis of V_1. The dual bases of V_0^* and V_1^* will be denoted by (e^i) and (θ_j).

Proposition 8. *With respect to the basis* $b = (e^1, \ldots, e^m, \theta_1, \ldots, \theta_n)$ *of* $V^* \cong V_0^* + V_1^*$ *the super Lie algebra* $\alpha(\mathfrak{p}(V)) \subset \mathfrak{gl}(V^*)$ *is identified with*

$$\alpha(\mathfrak{p}(V))^b = \left\{ \begin{pmatrix} A & 0 \\ C & \sigma(A) \end{pmatrix} \middle| A \in \mathfrak{so}(k, l),\ C^{ji} = e^i(\pi(s \vee \theta^j)), s \in V_1 \right\},$$

where $C = (C^{ji})$, $j = 1, \ldots, n$, $i = 1, \ldots, m$, *and* $\sigma : \mathfrak{so}(k, l) \to \mathfrak{gl}_n$ *is equivalent to the spinor representation.*

2.3. The (pseudo) Riemannian supergeometry associated to the spinor bundle

Now we carry over the construction of Section 2.2 to the $\mathbb{Z}_2$-graded vector bundle $V := TM_0 + S$ over M_0. We assume that M_0 is simply connected. The vector bundle V carries the canonical connection induced by the Levi-Civita connection of the pseudo-Riemannian manifold (M_0, g_0). The holonomy algebra of V at $p \in M_0$ is a subalgebra of $\mathfrak{spin}(T_pM_0) \subset \mathfrak{gl}(V_p)_0$. This implies, in particular, that the bundle of Spin(TM_0)-invariant bilinear forms on S is flat. Let g_1 be a parallel non-degenerate suitable bilinear form on S, see Definition 14 and the remarks following Theorem 1.

The Spin(TM_0)-invariant bilinear form $g = g_0 + g_1$ on V should be thought of as a pseudo-Riemannian metric for the supermanifold $M = M(S)$. Note that, due to Proposition 4, $g(p)$ induces a non-degenerate bilinear form on T_pM. However, recall that g_1 is symmetric or skew-symmetric. The map $\pi = \pi_{p,g_1} : \vee^2 S \to TM_0$ defines on $\mathfrak{p}(V) = \mathfrak{spin}(TM_0) + S \subset \mathfrak{gl}(V)$ the structure of bundle of Poincaré super Lie algebras. $\mathfrak{p}(V)$ is a parallel bundle. Now let $\alpha : \mathfrak{p}(V) \to \mathfrak{gl}(V^*)$ be the field of representations induced by the coadjoint representation, cf. Proposition 6. The image $\alpha(\mathfrak{p}(V)) \subset \mathfrak{gl}(V^*)$ is a parallel bundle of super Lie algebras.

Proposition 9. *The frame bundle of $V^* \to M$ has a subbundle P_{Spin_σ} with structure group $Spin_\sigma \subset GL_{m|n}(\mathbb{R})$, $Spin_\sigma \cong Spin(k,l)$, such that for all $b = (e^i, \theta_j) \in (P_{\mathrm{Spin}_\sigma})_p$:*
(1) *(e^i) is an orthonormal basis of $T_p^*M_0$ and*
(2) *$\alpha(\mathfrak{p}(V_p))$ is identified via b with the subalgebra $\mathfrak{g} = \alpha(\mathfrak{p}(V_p))^b \subset \mathfrak{gl}_{m|n}(\mathbb{R})$, where*

$$\mathfrak{g}_0 = \mathfrak{spin}_\sigma = \left\{ \begin{pmatrix} A & 0 \\ 0 & \sigma(A) \end{pmatrix} \middle| A \in \mathfrak{so}(k,l) \right\}$$

and

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \middle| C = (C^{ji}),\ C^{ji} = e^i(\pi(s \vee \theta^j)),\ s \in S_p \right\}$$

are independent of b and p. Here (θ^j) is the basis of S_p dual to (θ_j).

Proof. This follows from the holonomy reduction and Propositions 7 and 8. □

We denote by $\mathcal{V}$ the sheaf of local sections of V. Identifying TM_0 and T^*M_0 via g_0, the map ι of Proposition 5 corresponds to a monomorphism $\iota : \mathcal{V} = \mathcal{T}^*_{M_0} + \mathcal{S}^* \hookrightarrow \mathcal{T}_M$. This induces a map

$$\iota : \Gamma(U, P_{\mathrm{Spin}_\sigma}) \to \mathcal{F}(U),$$

where $\mathcal{F}(U)$ is the set of frame fields of M over the open set $U \subset M_0$. The image of ι generates a Spin$_\sigma$-structure on M, where Spin$_\sigma$ is now considered as (purely even) linear supergroup Spin$_\sigma \subset GL_{m|n}$. More precisely, recall that Spin$_\sigma(\mathcal{A}(U))$ is the group generated by $\exp \mathfrak{spin}_\sigma(\mathcal{A}(U)) \subset GL_{m|n}(\mathcal{A}(U))$. It acts on $\mathcal{F}(U)$ from the right. Put

$$\mathcal{F}_{\mathrm{Spin}_\sigma}(U) := \iota(\Gamma(U, P_{\mathrm{Spin}_\sigma}))\mathrm{Spin}_\sigma(\mathcal{A}(U)).$$

Proposition 10. *$\mathcal{F}_{\mathrm{Spin}_\sigma}$ is a $Spin_\sigma$-structure on M.*

Denote by G the linear supergroup defined by the linear super Lie algebra $\mathfrak{g}$, see Example 3. Since $\mathfrak{spin}_\sigma \subset \mathfrak{g} \subset \mathfrak{gl}_{m|n}(\mathbb{R})$, we have the following inclusions of linear supergroups:

$$\mathrm{Spin}_\sigma \subset G \subset GL_{m|n}. \tag{5}$$

Put $\mathcal{F}_G(U) := \mathcal{F}_{\mathrm{Spin}_\sigma}(U)G(\mathcal{A}(U))$ for all open $U \subset M_0$.

Proposition 11. *$\mathcal{F}_G$ is a G-structure on M.*

Definition 15. A *Killing vector field* on (M, g) is an infinitesimal automorphism of the G-structure $\mathcal{F}_G$, see Definition 12.

2.4. Twistor spinors as Killing vector fields

Definition 16. A section s of the spinor bundle $S \to M_0$ is called a *twistor spinor* if there exists a section $\tilde{s}$ of S such that

$$\nabla_X s = \rho(X)\tilde{s} \tag{6}$$

for all vector fields X on M_0. Here $\rho(X) : S \to S$ is Clifford multiplication. A twistor spinor s is called a Killing spinor if $\tilde{s} = \lambda s$ for some constant $\lambda \in \mathbb{R}$

Remark. From (6) it follows that $\tilde{s} = -(1/m)Ds$, where D is the Dirac operator.

The non-degenerate bilinear form g_1 on S induces the isomorphism

$$S \xrightarrow{\sim} S^*, \quad s \mapsto s^* := g_1(s, \cdot).$$

Recall that $\iota|\mathcal{S}^* : \mathcal{S}^* \hookrightarrow \mathcal{T}_M$ is simply given by interior multiplication, Section 2.1. To any spinor field S we associate the odd vector field $X_s := \iota(s^*)$ on M. Now we can state the main result of this paper.

Theorem 2. *Let (M_0, g_0) be a pseudo-Riemannian spin manifold with spinor bundle (S, g_1); g_1 a parallel non-degenerate suitable bilinear form on S, see Definition* 14 *and Section* 2.3. *Consider the supermanifold $M = M(S)$ with the bilinear form $g = g_0 + g_1$ and let s be a section of S. The vector field X_s is a Killing vector field on (M, g) iff s is a twistor spinor, see Definitions* 15 *and* 16.

Corollary 1. *A Killing vector field X_s for an extension g of g_0 is a Killing vector field for any other extension; the extensions being as in Section* 2.3.

Lemma 1. *For all sections s^*, t^* of S^* and X of TM_0 we have:*
(i) $[\iota(s^*), \iota(t^*)] = 0$,
(ii) $[\iota(s^*), \iota(X)] = [\iota(s^*), \nabla_X] = -\iota((\nabla_X)^*)$.

Proof.

(i) By definition of the supercommutator $[\cdot,\cdot]$ on $\mathcal{T}_M$, we have $[\iota(s^*), \iota(t^*)] = \iota(s^*) \circ \iota(t^*) + \iota(t^*) \circ \iota(s^*) = 0$.

(ii) Recall that $s^* = g_1(s,\cdot)$. If t is a section of S we have $[\iota(s^*), \iota(X)](t) = s^*(\nabla_X t) - \nabla_X s^*(t) = g_1(s, \nabla_X t) - \nabla_X g_1(s,t) = -g_1(\nabla_X s, t) = -(\nabla_X s)^*(t)$.

□

Proposition 12. *Let s be a twistor spinor. For all vector fields X and spinor fields t on M_0 we have:*

(i) $[\iota(s^*), \iota(X)] = -\iota((\rho(X)\tilde{s})^*) = -\tau(g_1)\iota(\rho(X)^*\tilde{s}^*)$, *where $\tau(g_1) \in \{\pm 1\}$ is the type of g_1, see Definition* 14.

(ii) $[\iota(s^*), \iota(X)](t) = -g_1(\rho(X)\tilde{s}, t) = -g_0(\pi(\tilde{s} \vee t), X)$.

Proof. The first equation of (i) follows from Lemma 1(ii), since $\nabla_X s = \rho(X)\tilde{s}$. Now the second equation of (i) and the first equation of (ii) follow from the definition of the type τ: $(\rho(X)\tilde{s})^*(t) = g_1(\rho(X)\tilde{s}, t) = \tau(g_1) g_1(\tilde{s}, \rho(X)t)$. The last equation of (ii) is simply the definition of $\pi = \pi_{\rho, g_1}$, cf. (4). □

Proof of Theorem 2. Let $(e^i, \theta_j) \in \Gamma(U, P_{\mathrm{Spin}_\sigma})$, $U \subset M_0$ open, and (e_i, θ^j) the dual local frame for $V = TM_0 + S$. Put

$$E := (\iota(e^i), \iota(\theta_j)) \in \Gamma(U, \mathcal{F}_{\mathrm{Spin}_\sigma}) \subset \Gamma(U, \mathcal{F}_G).$$

Since (e_i) is orthonormal, i.e. $g_0(e_i, e_j) = \varepsilon_i \delta_{ij}$, $\varepsilon_i \in \{\pm 1\}$, we have $e^i = \varepsilon_i g_0(e_i, \cdot)$. Hence, by definition of ι on $\mathcal{T}^*_{M_0}$, we have $\iota(e^i) = \varepsilon_i \iota(e_i)$. Therefore by Lemma 1 for any $s \in \Gamma(U, S)$ we have

$$L_{X_s} E = ([X_s, \iota(e^i)], [X_s, \iota(\theta_j)]) = (-\varepsilon_i \iota((\nabla_{e_i} s)^*), 0), \tag{7}$$

$$(\nabla_{e_i} s)^*(\theta^j) = g_1(\nabla_{e_i} s, \theta^j). \tag{8}$$

From this computation it follows that $L_{X_s} E \in E(\mathfrak{g} \otimes \mathcal{A}(U))$ iff there exists a $t \in \Gamma(U, S)$ such that

$$L_{X_s} E = E C_t, \tag{9}$$

where

$$C_t = \begin{pmatrix} 0 & 0 \\ (C_t^{ji}) & 0 \end{pmatrix} \in \mathfrak{g} \otimes \mathcal{A}(U), \quad C_t^{ji} = e^i(\pi(t \vee \theta^j)), \tag{10}$$

see Proposition 8. By (7), (8), and (10) Eq. (9) is equivalent to

$$g_1(\nabla_{e_i} s, \theta^j) = -\varepsilon_i e^i(\pi(t \vee \theta^j)), \quad i = 1, \ldots, m, \quad j = 1, \ldots, n. \tag{11}$$

The right-hand side is

$$-\varepsilon_i e^i(\pi(t \vee \theta^j)) = -g_0(\pi(t \vee \theta^j), e_i) = -g_1(\rho(e_i)t, \theta^j), \tag{12}$$

hence (11) is equivalent to the twistor equation (6) with $\tilde{s} = -t$. □

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